Upper Semicontinuity of Random Attractors for Non-compact Random Dynamical Systems

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Abstract

The upper semicontinuity of random attractors for non-compact random dynamical systems is proved when the union of all perturbed random attractors is precompact with probability one. This result is applied to the stochastic Reaction-Diffusion with white noise defined on the entire space $\mathbb{R}^n$.

Key words. Stochastic Reaction-Diffusion equation, random attractor, asymptotic compactness, upper semicontinuity.

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1 Introduction

In this paper, we study the limiting behavior of random attractors of non-compact random dynamical systems as stochastic perturbations approach zero. In particular, we will establish the upper semicontinuity of random attractors for the stochastically perturbed Reaction-Diffusion equation defined on the entire space $\mathbb{R}^n$:

$$du + (\lambda u - \Delta u)dt = (f(x,u) + g(x))dt + \epsilon h dW, \tag{1.1}$$

where ϵ is a small positive parameter, λ is a fixed positive constant, g and h are given functions defined on $\mathbb{R}^n$, f is a smooth nonlinear function satisfying some conditions, and W is a two-sided real-valued Wiener process on a complete probability space.

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By a random attractor we mean a compact and invariant random set which attracts all solutions when initial times approach minus infinity. The concept of random attractor was introduced in [12, 13] as extension to stochastic systems of the concept of global attractor for deterministic equations found in [2, 14, 21, 24, 25], for instance. In the case of bounded domains, random attractors for stochastic PDEs have been studied by many authors, see, e.g., [6, 7, 8, 9, 10, 11, 12, 13, 17, 18, 19, 20, 22, 23, 27, 28] and the references therein. In these papers, the asymptotic compactness of random dynamical systems follows directly from the compactness of Sobolev embeddings in bounded domains. This is the key to prove the existence of random attractors for PDEs defined in bounded domains. Since Sobolev embeddings are not compact on unbounded domains, the random dynamical systems associated with PDEs in this case are non-compact, and the asymptotic compactness of solutions cannot be obtained simply from these embeddings. This is a reason why there are only a few results on existence of random attractors for PDEs defined on unbounded domains. Nevertheless, the existence of such attractors for some stochastic PDEs on unbounded domains has been proved in [4, 26] recently. The asymptotic compactness and existence of absorbing sets for the stochastic Navier-Stokes equations on unbounded domains were established in [5].

In this paper, we will examine the limiting behavior of random attractors for the stochastically perturbed Reaction-Diffusion equation (1.1) defined on $\mathbb{R}^n$ when $\epsilon \to 0$, and prove the upper semicontinuity of these perturbed random attractors. In the deterministic case, the upper semicontinuity of global attractors were investigated in [14, 15, 16, 25] and many others. For stochastic PDEs defined in bounded domains, this problem has been studied by the authors of [9, 10, 19, 20, 22]. To the best of our knowledge, there is no result reported in the literature on the upper semicontinuity of random attractors for stochastic PDEs defined on unbounded domains. The purpose of this paper is to prove such a result for equation (1.1) on $\mathbb{R}^n$. Of course, the main difficulty here is the non-compactness of Sobolev embeddings on $\mathbb{R}^n$. In this paper, we will overcome the obstacles caused by the non-compactness of embeddings by using uniform estimates for far-field values of functions inside the perturbed random attractors. Actually, by a cut-off technique, we will show that the values of all functions in all perturbed random attractors are uniformly convergent to zero (in a sense) when spatial variables approach infinity (see the proof of Lemma 6.1 for more details).

The outline of this paper is as follows. We recall the basic random attractors theory in the

next section, and prove a result on the upper semicontinuity of random attractors in Section 3. This result works for non-compact random dynamical systems corresponding to stochastic PDEs defined on unbounded domains. In Section 4, we define a continuous random dynamical system for equation (1.1) in $L^2(\mathbb{R}^n)$. The uniform estimates of solutions for the equation are given in Section 5. Finally, we prove the upper semicontinuity of random attractors for (1.1) in the last section.

We denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and the inner product in $L^2(\mathbb{R}^n)$ and use $\| \cdot \|_p$ to denote the norm in $L^p(\mathbb{R}^n)$. Otherwise, the norm of a general Banach space X is written as $\| \cdot \|_X$. The letters c and c_i $(i = 1, 2, \ldots)$ are generic positive constants which may change their values from line to line or even in the same line.

2 Random attractors

We recall some basic concepts related to random attractors for stochastic dynamical systems. The reader is referred to [1, 3, 11, 13] for more details.

Let $(X, \| \cdot \|_X)$ be a Banach space with Borel σ-algebra $\mathcal{B}(X)$, and let $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 2.1. $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$ and $\theta_t P = P$ for all $t \in \mathbb{R}$.

Definition 2.2. A continuous random dynamical system (RDS) on X over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is a mapping

$$\phi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, x) \mapsto \phi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies, for P-a.e. $\omega \in \Omega$,

(i) $\phi(0, \omega, \cdot)$ is the identity on X;

(ii) $\phi(t + s, \omega, \cdot) = \phi(t, \theta_s \omega, \cdot) \circ \phi(s, \omega, \cdot)$ for all $t, s \in \mathbb{R}^+$;

(iii) $\phi(t, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}^+$.

Hereafter, we always assume that ϕ is a continuous RDS on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$.

Definition 2.3. A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for P-a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t}\|B(\theta_{-t}\omega)\|_X = 0 \quad \text{for all } \beta > 0,$$

where $\|B\|_X = \sup_{x\in B} \|x\|_X$.

Definition 2.4. Let $\mathcal{D}$ be a collection of random subsets of X. Then $\mathcal{D}$ is called inclusion-closed if $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\tilde{D} = \{\tilde{D}(\omega)\}_{\omega\in\Omega}$ with $\tilde{D}(\omega) \subseteq D(\omega)$ for all $\omega \in \Omega$ imply that $\tilde{D} \in \mathcal{D}$.

Definition 2.5. Let $\mathcal{D}$ be a collection of random subsets of X and $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called a random absorbing set for ϕ in $\mathcal{D}$ if for every $B \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $T(B,\omega) > 0$ such that

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \quad \text{for all } t \geq T(B,\omega).$$

Definition 2.6. Let $\mathcal{D}$ be a collection of random subsets of X. Then ϕ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\{\phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

Definition 2.7. Let $\mathcal{D}$ be a collection of random subsets of X. Then a random set $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ of X is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for ϕ if the following conditions are satisfied, for P-a.e. $\omega \in \Omega$,

(i) $\mathcal{A}(\omega)$ is compact, and $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$;

(ii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant, that is,

$$\phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega), \quad \forall\ t \geq 0;$$

(iii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ attracts every set in $\mathcal{D}$, that is, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$,

$$\lim_{t\to\infty} dist(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathcal{A}(\omega)) = 0,$$

where $dist(\cdot,\cdot)$ is the Hausdorff semi-metric given by $dist(Y,Z) = \sup_{y\in Y} \inf_{z\in Z} \|y - z\|_X$ for any $Y \subseteq X$ and $Z \subseteq X$.

The following existence result for a random attractor for a continuous RDS can be found in [3, 4, 13].

Proposition 2.8. *Let $\mathcal{D}$ be an inclusion-closed collection of random subsets of X and ϕ a continuous RDS on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. Suppose that $\{K(\omega)\}_{\omega\in K}$ is a closed random absorbing set for ϕ in $\mathcal{D}$ and ϕ is $\mathcal{D}$-pullback asymptotically compact in X. Then ϕ has a unique $\mathcal{D}$-random attractor $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ which is given by*

$$\mathcal{A}(\omega) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}.$$

3 Upper semicontinuity of random attractors

In this section, we establish the upper semicontinuity of random attractors when small random perturbations approach zero. Let $(X, \|\cdot\|_X)$ be a Banach space and Φ be an autonomous dynamical system defined on X. Given $\epsilon > 0$, suppose Φ_ϵ is a random dynamical system over a metric system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$. We further suppose that for P-a.e. $\omega \in \Omega$, $t \geq 0$, $\epsilon_n \to 0$, and x_n, $x \in X$ with $x_n \to x$, the following holds:

$$\lim_{n\to\infty} \Phi_{\epsilon_n}(t, \omega, x_n) = \Phi(t)x. \tag{3.1}$$

Let $\mathcal{D}$ be a collection of subsets of X. Given $\epsilon > 0$, suppose that Φ_ϵ has a random attractor $\mathcal{A}_\epsilon = \{\mathcal{A}_\epsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and a random absorbing set $E_\epsilon = \{E_\epsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ such that for some deterministic positive constant c and for P-a.e. $\omega \in \Omega$,

$$\limsup_{\epsilon\to 0} \|E_\epsilon(\omega)\|_X \leq c, \tag{3.2}$$

where $\|E_\epsilon(\omega)\|_X = \sup_{x\in E_\epsilon(\omega)} \|x\|_X$. We also assume that there exists $\epsilon_0 > 0$ such that for P-a.e. $\omega \in \Omega$,

$$\bigcup_{0<\epsilon\leq\epsilon_0} \mathcal{A}_\epsilon(\omega) \quad \text{is precompact in} \quad X. \tag{3.3}$$

Let $\mathcal{A}_0$ be the global attractor of Φ in X, which means that $\mathcal{A}_0$ is compact and invariant and attracts every bounded subset of X uniformly. Then the relationships between $\mathcal{A}_\epsilon$ and $\mathcal{A}_0$ are given by the following theorem.

Theorem 3.1. *Suppose* (3.1)-(3.3) *hold. Then for P-a.e. $\omega \in \Omega$,*

$$dist(\mathcal{A}_\epsilon(\omega), \mathcal{A}_0) \to 0, \quad as \quad \epsilon \to 0. \tag{3.4}$$

Proof. We argue by contradiction. If (3.4) is not true, then there $\delta > 0$ and a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in \mathcal{A}_{\epsilon_n}(\omega)$ and $\epsilon_n \to 0$ such that

$$dist(x_n, \mathcal{A}_0) \geq \delta. \tag{3.5}$$

It follows from (3.3) that there are $y_0 \in X$ and a subsequence of $\{x_n\}_{n=1}^{\infty}$ (still denoted by $\{x_n\}_{n=1}^{\infty}$) such that

$$\lim_{n\to\infty} x_n = y_0. \tag{3.6}$$

Next we prove $y_0 \in \mathcal{A}_0$. To this end, we take a sequence $\{t_m\}_{m=1}^{\infty}$ with $t_m \to \infty$. By the invariance of $\mathcal{A}_{\epsilon_n}$ we find that there exists a sequence $\{x_{1,n}\}_{n=1}^{\infty}$ with $x_{1,n} \in \mathcal{A}_{\epsilon_n}(\theta_{-t_1}\omega)$ such that

$$x_n = \Phi_{\epsilon_n}(t_1, \theta_{-t_1}\omega, x_{1,n}), \ \forall \ n \geq 1. \tag{3.7}$$

By (3.3) again, there exist $y_1 \in X$ and a subsequence of $\{x_{1,n}\}_{n=1}^{\infty}$ (still denoted by $\{x_{1,n}\}_{n=1}^{\infty}$) such that

$$\lim_{n\to\infty} x_{1,n} = y_1. \tag{3.8}$$

By (3.1) and (3.8) we find that

$$\lim_{n\to\infty} \Phi_{\epsilon_n}(t_1, \theta_{-t_1}\omega, x_{1,n}) = \Phi(t_1)y_1. \tag{3.9}$$

It follows from (3.6)-(3.7) and (3.9) that

$$y_0 = \Phi(t_1)y_1. \tag{3.10}$$

Since $x_{1,n} \in \mathcal{A}_{\epsilon_n}(\theta_{-t_1}\omega)$ and $\mathcal{A}_{\epsilon_n}(\theta_{-t_1}\omega) \subseteq E_{\epsilon_n}(\theta_{-t_1}\omega)$, by (3.2) we get

$$\limsup_{n\to\infty} \|x_{1,n}\|_X \leq \limsup_{n\to\infty} \|E_{\epsilon_n}(\theta_{-t_1}\omega)\|_X \leq c. \tag{3.11}$$

By (3.8) and (3.11) we find that

$$\|y_1\|_X \leq c.$$

Similarly, for each $m \geq 2$, repeating the above procedure, we can find that there is $y_m \in X$ such that

$$y_0 = \Phi(t_m)y_m, \quad \forall \ m \geq 2, \tag{3.12}$$

and

$$\|y_m\|_X \leq c, \quad \forall\ m \geq 2. \tag{3.13}$$

Since $t_m \to \infty$, (3.12) and (3.13) imply that $y_0 \in \mathcal{A}_0$. Therefore, by (3.6) we have

$$dist(x_n, \mathcal{A}) \leq dist(x_n, y_0) \to 0,$$

a contradiction with (3.5). This completes the proof. $\square$

We remark that the upper semicontinuity of random attractors for stochastic PDEs as perturbations of autonomous, non-autonomous and random systems was first proved by the authors in [9], [10] and [22], respectively. The conditions (3.1)-(3.3) of this paper are close but different from that given in [9, 10, 22]. For instance, the following condition is essentially assumed in [9, 10, 22] (see Theorem 2 on page 1562 in [9], Theorem 3.1 on page 496 in [10], and Theorem 2 on page 655 in [22]): there exists a compact set K such that, P-a.s.

$$\lim_{\epsilon \to 0} dist(\mathcal{A}_\epsilon(\omega), K) = 0. \tag{3.14}$$

For parabolic PDEs defined in *bounded* domains, the solution operators are compact, which follows from the regularity of solutions and the compactness of Sobolev embeddings. In that case, the existence of the compact set K satisfying condition (3.14) can be obtained by the existence of bounded absorbing sets in a space with higher regularity (see [9, 10, 22]). However, this method does not work for PDEs defined on *unbounded* domains because Sobolev embeddings are no longer compact in this case. Therefore, in the case of unbounded domains, it is difficult to find a compact set K which satisfies (3.14). In this paper, we require condition (3.3) rather than (3.14). As proved in Section 6 of this paper, the condition (3.3) is indeed fulfilled for the parabolic equation (1.1) defined on the unbounded domain $\mathbb{R}^n$, and hence the upper semicontinuity of the random attractors follows from Theorem 3.1 immediately.

4 Stochastic Reaction-Diffusion equations on $\mathbb{R}^n$

In this paper, we will investigate the upper semicontinuity of random attractors of the stochastic Reaction-Diffusion equation defined on $\mathbb{R}^n$. Given a small positive parameter ϵ, consider the

following stochastically perturbed equation:

$$du + (\lambda u - \Delta u)dt = (f(x, u) + g(x))dt + \epsilon h\, dW, \quad x \in \mathbb{R}^n, \quad t > 0, \tag{4.1}$$

with the initial condition:

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n. \tag{4.2}$$

Here ϵ and λ are positive constants, g is a given function in $L^2(\mathbb{R}^n)$, $h \in H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$ for some $p \geq 2$, W is a two-sided real-valued Wiener process on a complete probability space $(\Omega, \mathcal{F}, P)$, where P is the Wiener distribution, Ω is a subset of $\{\omega \in C(\mathbb{R}, \mathbb{R}) : \ \omega(0) = 0\}$ with $P(\Omega) = 1$, and $\mathcal{F}$ is a σ-algebra. In addition, the space $(\Omega, \mathcal{F}, P)$ is invariant under the Wiener shift:

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, \quad t \in \mathbb{R}.$$

This means that $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system (see, e.g., [9, 23] for existence of this space).

Consider the one-dimensional Ornstein-Uhlenbeck equation:

$$dy + \lambda y dt = dW(t). \tag{4.3}$$

One may easily check that a solution to (4.3) is given by

$$y(\theta_t \omega) = -\lambda \int_{-\infty}^{0} e^{\lambda \tau} (\theta_t \omega)(\tau) d\tau, \quad t \in \mathbb{R}.$$

Note that the random variable $|y(\omega)|$ is tempered and $y(\theta_t \omega)$ is P-a.e. continuous. Therefore, it follows from Proposition 4.3.3 in [1] that there exists a tempered function $r(\omega) > 0$ such that

$$|y(\omega)|^2 + |y(\omega)|^p \leq r(\omega), \tag{4.4}$$

where $r(\omega)$ satisfies, for P-a.e. $\omega \in \Omega$,

$$r(\theta_t \omega) \leq e^{\frac{\lambda}{2}|t|} r(\omega), \quad t \in \mathbb{R}. \tag{4.5}$$

Then it follows from (4.4)-(4.5) that, for P-a.e. $\omega \in \Omega$,

$$|y(\theta_t \omega)|^2 + |y(\theta_t \omega)|^p \leq e^{\frac{\lambda}{2}|t|} r(\omega), \quad t \in \mathbb{R}. \tag{4.6}$$

Let $z(\theta_t\omega) = hy(\theta_t\omega)$ and $v(t) = u(t) - \epsilon z(\theta_t\omega)$ where u is a solution of problem (4.1)-(4.2). Then v satisfies

$$\frac{\partial v}{\partial t} + \lambda v - \Delta v = f(x, v + \epsilon z(\theta_t\omega)) + g + \epsilon\Delta z(\theta_t\omega). \tag{4.7}$$

In this paper, we assume that the nonlinearity f satisfies the following conditions: For all $x \in R^n$ and $s \in R$,

$$f(x,s)s \leq -\alpha_1|s|^p + \psi_1(x), \tag{4.8}$$

$$|f(x,s)| \leq \alpha_2|s|^{p-1} + \psi_2(x), \tag{4.9}$$

$$\frac{\partial f}{\partial s}(x,s) \leq \beta, \tag{4.10}$$

$$|\frac{\partial f}{\partial x}(x,s)| \leq \psi_3(x), \tag{4.11}$$

where α_1, α_2 and β are positive constants, $\psi_1 \in L^1(R^n) \cap L^\infty(R^n)$, and $\psi_2 \in L^2(R^n) \cap L^q(\mathbb{R}^n)$ with $\frac{1}{q} + \frac{1}{p} = 1$, and $\psi_3 \in L^2(\mathbb{R}^n)$.

It follows from [4] that, under conditions (4.8)-(4.11), for P-a.e. $\omega \in \Omega$ and for all $v_0 \in L^2(\mathbb{R}^n)$, (4.7) has a unique solution $v(\cdot, \omega, v_0) \in C([0,\infty), L^2(\mathbb{R}^n)) \bigcap L^2((0,T), H^1(\mathbb{R}^n))$ with $v(0,\omega,v_0) = v_0$ for every $T > 0$. Furthermore , the solution is continuous with respect to v_0 in $L^2(\mathbb{R}^n)$ for all $t \geq 0$. Let

$$u(t,\omega,u_0) = v(t,\omega,v_0) + \epsilon z(\theta_t\omega), \quad \text{where} \quad v_0 = u_0 - \epsilon z(\omega). \tag{4.12}$$

We can associate a random dynamical system Φ_ϵ with problem (4.1)-(4.2) via u for each $\epsilon > 0$, where $\Phi_\epsilon : \mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is given by

$$\Phi_\epsilon(t,\omega,u_0) = u(t,\omega,u_0), \quad \text{for every} \quad (t,\omega,u_0) \in \mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n). \tag{4.13}$$

Then Φ_ϵ is a continuous random dynamical system over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ in $L^2(\mathbb{R}^n)$. In the sequel, we always assume that $\mathcal{D}$ is a collection of random subsets of $L^2(\mathbb{R}^n)$ given by

$$\mathcal{D} = \{D = \{D(\omega)\}_{\omega\in\Omega},\ D(\omega) \subseteq L^2(\mathbb{R}^n) \quad \text{and} \quad e^{-\frac{1}{2}\lambda t}\|B(\theta_{-t}\omega)\| \to 0 \text{ as } t \to \infty\}, \tag{4.14}$$

where

$$\|B(\theta_{-t}\omega)\| = \sup_{u\in B(\theta_{-t}\omega)} \|u\|.$$

In [4], the authors proved that Φ_ϵ has a $\mathcal{D}$-pullback random attractor if $\mathcal{D}$ is the collection of all tempered random subsets of $L^2(\mathbb{R}^n)$. Following the arguments of [4], we can also prove that Φ_ϵ has a unique $\mathcal{D}$-pullback random attractor $\{\mathcal{A}_\epsilon(\omega)\}_{\omega\in\Omega}$ when $\mathcal{D}$ is given by (4.14) (the existence of $\{\mathcal{A}_\epsilon(\omega)\}_{\omega\in\Omega}$ in this case is also implied by the estimates given in Section 5 of this paper). When $\epsilon = 0$, problem (4.1)-(4.2) defines a continuous deterministic dynamical system Φ in $L^2(\mathbb{R}^n)$. In this case, the results of [4] imply that Φ has a unique global attractor $\mathcal{A}$ in $L^2(\mathbb{R}^n)$. The purpose of this paper is to establish the relationships of $\{\mathcal{A}_\epsilon(\omega)\}_{\omega\in\Omega}$ and $\mathcal{A}$ when $\epsilon \to 0$.

5 Uniform estimates of solutions

In this section, we derive uniform estimates of solutions with respect to the small parameter ϵ. These estimates are useful for proving the semicontinuity of the perturbed random attractors. Here and after, we always assume that $\mathcal{D}$ is the collection of random subsets of $L^2(\mathbb{R}^n)$ given in (4.14).

Lemma 5.1. *Let $0 < \epsilon \le 1$, $g \in L^2(\mathbb{R}^n)$ and (4.8)-(4.11) hold. Then for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there is $T(B,\omega) > 0$, independent of ϵ, such that for all $v_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$,*

$$\|v(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 \le e^{-\lambda t}\|v_0(\theta_{-t}\omega)\|^2 + c + \epsilon c r(\omega), \quad \forall\ t \ge 0,$$

$$\int_0^t e^{\lambda(\tau - t)}\|\nabla v(\tau,\theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 d\tau \le e^{-\lambda t}\|v_0(\theta_{-t}\omega)\|^2 + c + \epsilon c r(\omega), \quad \forall\ t \ge 0,$$

and

$$\int_0^t e^{\lambda(\tau - t)}\|u(\tau,\theta_{-t}\omega, u_0(\theta_{-t}\omega))\|_p^p d\tau \le c + \epsilon c r(\omega), \quad \forall\ t \ge T(B,\omega),$$

where c is a positive deterministic constant independent of ϵ, and $r(\omega)$ is the tempered function in (4.4).

Proof. The idea of proof is similar to that given in [4], but now we have to pay attention to how the estimates depend on the parameter ϵ. Multiplying (4.7) by v and then integrating over $\mathbb{R}^n$, we find that

$$\frac{1}{2}\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 = \int_{\mathbb{R}^n} f(x, v + \epsilon z(\theta_t\omega))\ v dx + (g, v) + \epsilon(\Delta z(\theta_t\omega), v). \tag{5.1}$$

For the nonlinear term, by (4.8)-(4.9) we obtain

$$\int_{\mathbb{R}^n} f(x, v+\epsilon z(\theta_t\omega))\ v dx = \int_{\mathbb{R}^n} f(x, v+\epsilon z(\theta_t\omega))\ (v+\epsilon z(\theta_t\omega))dx - \epsilon \int_{\mathbb{R}^n} f(x, v+\epsilon z(\theta_t\omega))\ z(\theta_t\omega)dx$$

$$\le -\alpha_1 \int_{\mathbb{R}^n} |u|^p dx + \int_{\mathbb{R}^n} \psi_1(x)dx - \epsilon \int_{\mathbb{R}^n} f(x,u)\ z(\theta_t\omega)dx$$

$$\le -\frac{1}{2}\alpha_1 \|u\|_p^p + \epsilon c_2(\|z(\theta_t\omega)\|_p^p + \|z(\theta_t\omega)\|^2) + c_3, \tag{5.2}$$

where c_2 and c_3 do not depend on ϵ. Similarly, the remaining terms on the right-hand side of (5.1) are bounded by

$$\|g\|\|v\| + \epsilon\|\nabla z(\theta_t\omega)\|\|\nabla v\| \le \frac{1}{2}\lambda\|v\|^2 + \frac{1}{2\lambda}\|g\|^2 + \frac{1}{2}\epsilon\|\nabla z(\theta_t\omega)\|^2 + \frac{1}{2}\|\nabla v\|^2. \tag{5.3}$$

Then it follows from (5.1)-(5.3) that

$$\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 + \alpha_1\|u\|_p^p \le \epsilon c_4(\|z(\theta_t\omega)\|_p^p + \|z(\theta_t\omega)\|^2 + \|\nabla z(\theta_t\omega)\|^2) + c_5. \tag{5.4}$$

Note that $z(\theta_t\omega) = hy(\theta_t\omega)$ and $h \in H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$. Then we have

$$\|z(\theta_t\omega)\|_p^p + \|z(\theta_t\omega)\|^2 + \|\nabla z(\theta_t\omega)\|^2 \le c_6(|y(\theta_t\omega)|^p + |y(\theta_t\omega)|^2) = p_1(\theta_t\omega). \tag{5.5}$$

By (4.6), we find that for P-a.e. $\omega \in \Omega$,

$$p_1(\theta_\tau\omega) \le c_6 e^{\frac{1}{2}\lambda|\tau|} r(\omega), \quad \forall\ \tau \in \mathbb{R}. \tag{5.6}$$

It follows from (5.4)-(5.5) that, for all $t \ge 0$,

$$\frac{d}{dt}\|v\|^2 + \lambda\|v\|^2 + \|\nabla v\|^2 + \alpha_1\|u\|_p^p \le \epsilon c_4 p_1(\theta_t\omega) + c_5. \tag{5.7}$$

Multiplying (5.7) by $e^{\lambda t}$ and then integrating the inequality, we get that, for all $t \ge 0$,

$$\|v(t,\omega,v_0(\omega))\|^2 + \int_0^t e^{\lambda(\tau-t)}\|\nabla v(\tau,\omega,v_0(\omega))\|^2 d\tau + \alpha_1 \int_0^t e^{\lambda(\tau-t)}\|u(\tau,\omega,u_0(\omega))\|_p^p d\tau$$

$$\le e^{-\lambda t}\|v_0(\omega)\|^2 + \epsilon c_4 \int_0^t e^{\lambda(\tau-t)} p_1(\theta_\tau\omega)d\tau + c_7. \tag{5.8}$$

By replacing ω by $\theta_{-t}\omega$, we get from (5.8) and (5.6) that, for all $t \ge 0$,

$$\|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 + \int_0^t e^{\lambda(\tau-t)}\|\nabla v(\tau,\theta_{-t}\omega,v_0(\theta_{-t}\omega))\|^2 d\tau + \alpha_1 \int_0^t e^{\lambda(\tau-t)}\|u(\tau,\theta_{-t}\omega,u_0(\theta_{-t}\omega))\|_p^p d\tau$$

$$\leq e^{-\lambda t}\|v_0(\theta_{-t}\omega)\|^2 + \epsilon c_4 \int_0^t e^{\lambda(\tau-t)} p_1(\theta_{\tau-t}\omega) ds + c_7 \leq e^{-\lambda t}\|v_0(\theta_{-t}\omega)\|^2 + \epsilon c_9 r(\omega) + c_7. \quad (5.9)$$

Since $v_0(\theta_{-t}\omega) \in \mathcal{D}$, there is $T = T(B, \omega)$, independent of ϵ, such that for all $t \geq T$,

$$e^{-\lambda t}\|v_0(\theta_{-t}\omega)\|^2 \leq 1,$$

which along with (5.9) implies the lemma. $\square$

As a consequence of Lemma 5.1, we have the following estimates for u.

Lemma 5.2. *Let $0 < \epsilon \leq 1$, $g \in L^2(\mathbb{R}^n)$ and (4.8)-(4.11) hold. Then for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there is $T(B, \omega) > 0$, independent of ϵ, such that for all $t \geq T(B, \omega)$ and $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$,*

$$\|u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 \leq c + \epsilon c r(\omega),$$

and

$$\int_t^{t+1} \|\nabla u(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2 d\tau \leq c + \epsilon c r(\omega),$$

where c is a positive deterministic constant independent of ϵ, and $r(\omega)$ is the tempered function in (4.4).

Proof. It follows from (4.12) and Lemma 5.1 that

$$\|u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 \leq 2\|v(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega) - \epsilon z(\theta_{-t}\omega))\|^2 + 2\epsilon^2\|z(\omega)\|^2$$

$$\leq 4e^{-\lambda t}(\|u_0(\theta_{-t}\omega)\|^2 + \|z(\theta_{-t}\omega)\|^2) + c + \epsilon c r(\omega), \quad (5.10)$$

where we have used (4.4) and the fact $0 < \epsilon \leq 1$. Since $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$ and $\|z(\omega)\|^2$ is tempered, there is $T(B, \omega) > 0$, independent of ϵ, such that for all $t \geq T(B, \omega)$,

$$e^{-\lambda t}(\|u_0(\theta_{-t}\omega)\|^2 + \|z(\theta_{-t}\omega)\|^2) \leq 1, \quad (5.11)$$

which along with (5.10) implies that, for all $t \geq T(B, \omega)$,

$$\|u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 \leq 4 + c + \epsilon c r(\omega). \quad (5.12)$$

Similarly, we have

$$\|\nabla u(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2 = \|\nabla v(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega) - \epsilon z(\theta_{-t-1}\omega)) + \epsilon\nabla z(\theta_{\tau-t-1}\omega)\|^2$$

$$\leq 2\|\nabla v(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega) - \epsilon z(\theta_{-t-1}\omega))\|^2 + 2\epsilon^2 \|\nabla z(\theta_{\tau-t-1}\omega)\|^2 \tag{5.13}$$

For $\tau \in (t, t+1)$, by (4.6) we find that

$$\|\nabla z(\theta_{\tau-t-1}\omega)\|^2 \leq c|y(\theta_{\tau-t-1}\omega)|^2 \leq ce^{\frac{\lambda}{2}} r(\omega). \tag{5.14}$$

By (5.13) and (5.14), we get

$$\|\nabla u(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2 \leq 2\|\nabla v(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega) - \epsilon z(\theta_{-t-1}\omega))\|^2 + \epsilon c r(\omega).$$

Integrating the above with respect to τ in $(t, t+1)$ we obtain

$$\int_t^{t+1} \|\nabla u(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2 d\tau$$

$$\leq 2 \int_t^{t+1} \|\nabla v(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega) - \epsilon z(\theta_{-t-1}\omega))\|^2 d\tau + \epsilon c r(\omega). \tag{5.15}$$

Given $t \geq 0$, replacing t by $t+1$ in Lemma 5.1 we find that

$$\int_t^{t+1} e^{\lambda(\tau-t-1)} \|\nabla v(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega) - \epsilon z(\theta_{-t-1}\omega))\|^2 d\tau$$

$$\leq 2e^{-\lambda(t+1)}(\|u_0(\theta_{-t-1}\omega)\|^2 + \|z(\theta_{-t-1}\omega)\|^2) + c + \epsilon c r(\omega). \tag{5.16}$$

Replacing t by $t+1$ in (5.11), we find that the first term on the right-hand side of (5.16) is less than 2 when $t \geq T(B, \omega)$. Therefore, we have, for all $t \geq T(B, \omega)$,

$$\int_t^{t+1} e^{\lambda(\tau-t-1)} \|\nabla v(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega) - \epsilon z(\theta_{-t-1}\omega))\|^2 d\tau \leq 2 + c + \epsilon c r(\omega).$$

Since $e^{\lambda(\tau-t-1)} \geq e^{-\lambda}$ for $\tau \in (t, t+1)$, the above implies that, for all $t \geq T(B, \omega)$,

$$\int_t^{t+1} \|\nabla v(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega) - \epsilon z(\theta_{-t-1}\omega))\|^2 d\tau \leq e^{\lambda}(2 + c + \epsilon c r(\omega)). \tag{5.17}$$

It follows from (5.15) and (5.17) that, for all $t \geq T(B, \omega)$,

$$\int_t^{t+1} \|\nabla u(\tau, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2 d\tau \leq c + \epsilon c r(\omega),$$

which along with (5.12) concludes the proof. $\square$

We are now in a position to establish the uniform estimates of solutions in $H^1(\mathbb{R}^n)$.

Lemma 5.3. *Let $0 < \epsilon \le 1$, $g \in L^2(\mathbb{R}^n)$ and (4.8)-(4.11) hold. Then for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there is $T(B,\omega) > 0$, independent of ϵ, such that for all $t \ge T(B,\omega)$, $u_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega)$ and $v_0(\theta_{-t}\omega) = u_0(\theta_{-t}\omega) - \epsilon z(\omega)$,*

$$\|\nabla v(t, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 \le c + \epsilon c r(\omega),$$

and

$$\|\nabla u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 \le c + \epsilon c r(\omega),$$

where c is a positive deterministic constant independent of ϵ, and $r(\omega)$ is the tempered function in (4.4).

Proof. Taking the inner product of (4.7) with Δv in $L^2(\mathbb{R}^n)$, we get that

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|^2 + \lambda\|\nabla v\|^2 + \|\Delta v\|^2 = -\int_{\mathbb{R}^n} f(x,u)\Delta v dx - (g + \epsilon\Delta z(\theta_t\omega), \Delta v). \tag{5.18}$$

By (4.9)-(4.11), the first term on the right-hand side of (5.18) satisfies

$$-\int_{\mathbb{R}^n} f(x,u)\ \Delta v dx = -\int_{\mathbb{R}^n} f(x,u)\ \Delta u dx + \epsilon\int_{\mathbb{R}^n} f(x,u)\ \Delta z(\theta_t\omega) dx$$

$$= \int_{\mathbb{R}^n} \frac{\partial f}{\partial x}(x,u)\ \nabla u dx + \int_{\mathbb{R}^n} \frac{\partial f}{\partial u}(x,u)\ |\nabla u|^2 dx + \epsilon\int_{\mathbb{R}^n} f(x,u)\Delta z(\theta_t\omega) dx$$

$$\le c\left(\|\nabla u\|^2 + \|u\|_p^p\right) + \epsilon c\left(\|\Delta z(\theta_t\omega)\|^2 + \|\Delta z(\theta_t\omega)\|_p^p\right) + c, \tag{5.19}$$

where we have used the fact $0 < \epsilon \le 1$. For the last term on the right-hand side of (5.18), we have

$$|(g, \Delta v)| + \epsilon|(\Delta z(\theta_t\omega), \Delta v)| \le \frac{1}{2}\|\Delta v\|^2 + \|g\|^2 + \epsilon\|\Delta z(\theta_t\omega)\|^2. \tag{5.20}$$

It follows from (5.18)-(5.20) that, for all $t \ge 0$,

$$\frac{d}{dt}\|\nabla v\|^2 \le c\left(\|\nabla u\|^2 + \|u\|_p^p\right) + \epsilon c\left(\|\Delta z(\theta_t\omega)\|^2 + \|\Delta z(\theta_t\omega)\|_p^p\right) + c$$

$$\le c\left(\|\nabla u\|^2 + \|u\|_p^p\right) + \epsilon c p_2(\theta_t\omega) + c, \tag{5.21}$$

where $p_2(\theta_t\omega) = \|\Delta z(\theta_t\omega)\|^2 + \|\Delta z(\theta_t\omega)\|_p^p$. Let $T(B,\omega)$ be the constant in Lemma 5.2, fix $t \ge T(B,\omega)$ and $s \in (t, t+1)$. Integrating (5.21) in $(s, t+1)$ we find that

$$\|\nabla v(t+1, \omega, v_0(\omega))\|^2 \le \|\nabla v(s, \omega, v_0(\omega))\|^2 + \epsilon c\int_s^{t+1} p_2(\theta_\tau\omega) d\tau$$

$$+c\int_s^{t+1}\left(\|\nabla u(\tau,\omega,u_0(\omega))\|^2+\|u(\tau,\omega,u_0(\omega))\|_p^p\right)d\tau+c.$$

$$\le\|\nabla v(s,\omega,v_0(\omega))\|^2+\epsilon c\int_t^{t+1}p_2(\theta_\tau\omega)d\tau$$

$$+c\int_t^{t+1}\left(\|\nabla u(\tau,\omega,u_0(\omega))\|^2+\|u(\tau,\omega,u_0(\omega))\|_p^p\right)d\tau+c.$$

Integrating the above with respect to s in $(t,t+1)$, we have

$$\|\nabla v(t+1,\omega,v_0(\omega))\|^2\le\int_t^{t+1}\|\nabla v(s,\omega,v_0(\omega))\|^2ds+\epsilon c\int_t^{t+1}p_2(\theta_\tau\omega)d\tau$$

$$+c\int_t^{t+1}\left(\|\nabla u(\tau,\omega,u_0(\omega))\|^2+\|u(\tau,\omega,u_0(\omega))\|_p^p\right)d\tau+c.$$

Now replacing ω by $\theta_{-t-1}\omega$, we get that

$$\|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2$$

$$\le\int_t^{t+1}\|\nabla v(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2ds+\epsilon c\int_t^{t+1}p_2(\theta_{\tau-t-1}\omega)d\tau$$

$$+c\int_t^{t+1}\left(\|\nabla u(\tau,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|^2+\|u(\tau,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^p\right)d\tau+c. \tag{5.22}$$

Replacing t by $t+1$ in Lemma 5.1, we find that there exists $T_1=T_1(B,\omega)>0$, independent of ϵ, such that for all $t\ge T_1$,

$$\int_t^{t+1}e^{\lambda(\tau-t-1)}\|\nabla v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2d\tau\le c+\epsilon cr(\omega), \tag{5.23}$$

and

$$\int_t^{t+1}e^{\lambda(\tau-t-1)}\|u(\tau,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^pd\tau\le c+\epsilon cr(\omega). \tag{5.24}$$

Since $e^{\lambda(\tau-t-1)}\ge e^{-\lambda}$ for $\tau\in(t,t+1)$, we obtain from (5.23)-(5.24) that, for all $t\ge T_1$,

$$\int_t^{t+1}(\|\nabla v(\tau,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2+\|u(\tau,\theta_{-t-1}\omega,u_0(\theta_{-t-1}\omega))\|_p^p)d\tau\le ce^{\lambda}(1+\epsilon r(\omega)). \tag{5.25}$$

It follows from (5.22), (5.25) and Lemma 5.2 that, there is $T_2=T_2(B,\omega)>0$, independent of ϵ, such that for all $t\ge T_2$,

$$\|\nabla v(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega))\|^2\le c_1+\epsilon c_2r(\omega)+\epsilon c\int_{-1}^0p_2(\theta_\tau\omega)d\tau$$

$$\leq c_1 + \epsilon c_2 r(\omega) + \epsilon c_3 \int_{-1}^{0} e^{-\frac{\lambda}{2}\tau} r(\omega) d\tau \leq c_1 + \epsilon c_4 r(\omega), \tag{5.26}$$

where we have used (4.6). From (4.12) and (5.26) we have, for all $t \geq T_2$,

$$\|\nabla u(t+1, \theta_{-t-1}\omega, u_0(\theta_{-t-1}\omega))\|^2 \leq c_5 + \epsilon c_6 r(\omega). \tag{5.27}$$

The lemma then follows from (5.26) and (5.27). □

Next, we derive uniform estimates of solutions for large space and time variables. Particularly, we show how these estimates depend on the small parameter ϵ.

Lemma 5.4. *Let $0 < \epsilon \leq 1$, $g \in L^2(\mathbb{R}^n)$ and (4.8)-(4.11) hold. Suppose $B = \{B(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ and $u_0(\omega) \in B(\omega)$. Then for every $\eta > 0$ and P-a.e. $\omega \in \Omega$, there exist $T = T(B, \omega, \eta) > 0$ and $R = R(\omega, \eta) > 0$ such that for all $t \geq T$,*

$$\int_{|x| \geq R} |u(t, \theta_{-t}\omega, u_0(\theta_{-t}\omega))(x)|^2 dx \leq \eta,$$

where $T(B, \omega, \eta)$ and $R(\omega, \eta)$ do not depend on ϵ.

Proof. Let ρ be a smooth function defined on $\mathbb{R}^+$ such that $0 \leq \rho(s) \leq 1$ for all $s \in \mathbb{R}^+$, and

$$\rho(s) = \begin{cases} 0 & \text{for } 0 \leq s \leq 1; \\ 1 & \text{for } s \geq 2. \end{cases}$$

Then there exists a positive constant c such that $|\rho'(s)| \leq c$ for all $s \in \mathbb{R}^+$.

Taking the inner product of (4.7) with $\rho(\frac{|x|^2}{k^2})v$ in $L^2(\mathbb{R}^n)$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx + \lambda \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx + \int_{\mathbb{R}^n} |\nabla v|^2 \rho(\frac{|x|^2}{k^2}) dx$$

$$= \int_{\mathbb{R}^n} f(x,u)\rho(\frac{|x|^2}{k^2}) v dx - \int_{\mathbb{R}^n} v\rho'(\frac{|x|^2}{k^2})\frac{2x}{k^2} \cdot \nabla v dx + \int_{\mathbb{R}^n} (g + \epsilon \Delta z(\theta_t \omega))\, \rho(\frac{|x|^2}{k^2}) v dx. \tag{5.28}$$

By (4.8) and (4.9), the first term on the right-hand side of (5.28) satisfies

$$\int_{\mathbb{R}^n} f(x,u)\rho(\frac{|x|^2}{k^2}) v dx = \int_{\mathbb{R}^n} f(x,u)\rho(\frac{|x|^2}{k^2}) u dx - \epsilon \int_{\mathbb{R}^n} f(x,u)\rho(\frac{|x|^2}{k^2}) z(\theta_t\omega) dx$$

$$\leq -\frac{1}{2}\alpha_1 \int_{\mathbb{R}^n} |u|^p \rho(\frac{|x|^2}{k^2}) dx + \int_{\mathbb{R}^n} \psi_1 \rho(\frac{|x|^2}{k^2}) dx$$

$$+\frac{1}{2}\int_{\mathbb{R}^n}\psi_2^2\rho(\frac{|x|^2}{k^2})dx+\epsilon c\int_{\mathbb{R}^n}\left(|z(\theta_t\omega)|^p+|z(\theta_t\omega)|^2\right)\rho(\frac{|x|^2}{k^2})dx. \tag{5.29}$$

Note that the second term on the right-hand side of (5.28) is bounded by

$$|\int_{\mathbb{R}^n} v\rho'(\frac{|x|^2}{k^2})\frac{2x}{k^2}\cdot\nabla v dx| = |\int_{k\le|x|\le\sqrt{2}k} v\rho'(\frac{|x|^2}{k^2})\frac{2x}{k^2}\cdot\nabla v dx|$$

$$\le\frac{2\sqrt{2}}{k}\int_{k\le|x|\le\sqrt{2}k}|v|\ |\rho'(\frac{|x|^2}{k^2})|\ |\nabla v|dx\le\frac{c}{k}(\|v\|^2+\|\nabla v\|^2). \tag{5.30}$$

For the last term on the right-hand side of (5.28), we have

$$|\int_{\mathbb{R}^n}(g+\epsilon\Delta z(\theta_t\omega))\rho(\frac{|x|^2}{k^2})v dx|\le\frac{1}{2}\lambda\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx+\frac{1}{\lambda}\int_{\mathbb{R}^n}(g^2+\epsilon^2|\Delta z(\theta_t\omega)|^2)\rho(\frac{|x|^2}{k^2})dx. \tag{5.31}$$

It follows from (5.28)-(5.31) that

$$\frac{d}{dt}\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx+\lambda\int_{\mathbb{R}^n}\rho(\frac{|x|^2}{k^2})|v|^2dx$$

$$\le\frac{c}{k}(\|\nabla v\|^2+\|v\|^2)+c\int_{\mathbb{R}^n}\left(|\psi_1|+|\psi_2|^2+g^2\right)\rho(\frac{|x|^2}{k^2})dx$$

$$+\epsilon c\int_{\mathbb{R}^n}\left(|\Delta z(\theta_t\omega)|^2+|z(\theta_t\omega)|^2+|z(\theta_t\omega)|^p\right)\rho(\frac{|x|^2}{k^2})dx. \tag{5.32}$$

Then using Lemmas 5.1-5.3 and following the process of [4], after detailed calculations we find that, given $\eta>0$, there exist $T=T(B,\omega,\eta)$ and $R=R(B,\eta)$, which are independent of ϵ, such that for all $t\ge T$ and $k\ge R$,

$$\int_{|x|\ge k}|v(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega))|^2dx\le\eta,$$

which along with (4.12) implies the lemma. $\square$

6 Upper semicontinuity of random attractors for Reaction-Diffusion equations on $\mathbb{R}^n$

In this section, we prove the upper semicontinuity of random attractors for the Reaction-Diffusion equation defined on $\mathbb{R}^n$ when the stochastic perturbations approach zero. To this end, we first establish the convergence of solutions of problem (4.1)-(4.2) when $\epsilon\to 0$, and then show that the union of all perturbed random attractors is precompact in $L^2(\mathbb{R}^n)$.

To indicate dependence of solutions on ϵ, in this section, we write the solution of problem (4.1)-(4.2) as u^ϵ, and the corresponding cocycle as Φ_ϵ. Given $0 < \epsilon \le 1$, it follows from Lemma 5.2 that, for every $B = \{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $T = T(B, \omega) > 0$, independent of ϵ, such that for all $t \ge T$,

$$\|\Phi_\epsilon(t, \theta_{-t}\omega, B(\theta_{-t}\omega))\| \le M + \epsilon M r(\omega), \tag{6.1}$$

where M is a positive deterministic constant independent of ϵ, and $r(\omega)$ is the tempered function in (4.4). Denote by

$$K_\epsilon(\omega) = \{u \in L^2(\mathbb{R}^n) : \ \|u\| \le M + \epsilon M r(\omega)\}, \tag{6.2}$$

and

$$K(\omega) = \{u \in L^2(\mathbb{R}^n) : \ \|u\| \le M + M r(\omega)\}, \tag{6.3}$$

where M is the constant in (6.1). Then for every $0 < \epsilon \le 1$, $\{K_\epsilon(\omega)\}_{\omega\in\Omega}$ is a closed absorbing set for Φ_ϵ in $\mathcal{D}$ and

$$\bigcup_{0<\epsilon\le 1} K_\epsilon(\omega) \subseteq K(\omega). \tag{6.4}$$

It follows from the invariance of the random attractor $\{\mathcal{A}_\epsilon(\omega)\}_{\omega\in\Omega}$ and (6.4) that

$$\bigcup_{0<\epsilon\le 1} \mathcal{A}_\epsilon(\omega) \subseteq \bigcup_{0<\epsilon\le 1} K_\epsilon(\omega) \subseteq K(\omega). \tag{6.5}$$

On the other hand, by Lemmas 5.2 and 5.3, we find that, for every $0 < \epsilon \le 1$ and P-a.e. $\omega \in \Omega$, there exists $T_1 = T_1(\omega) > 0$, independent of ϵ, such that for all $t \ge T_1$,

$$\|\Phi_\epsilon(t, \theta_{-t}\omega, K(\theta_{-t}\omega))\|_{H^1(\mathbb{R}^n)} \le M_1 + \epsilon M_1 r(\omega) \le M_1 + M_1 r(\omega), \tag{6.6}$$

where $K(\omega)$ is given in (6.3) and M_1 is a positive deterministic constant independent of ϵ. By (6.5) and (6.6) we obtain that, for every $0 < \epsilon \le 1$, P-a.e. $\omega \in \Omega$ and $t \ge T_1$,

$$\|\Phi_\epsilon(t, \theta_{-t}\omega, \mathcal{A}_\epsilon(\theta_{-t}\omega))\|_{H^1(\mathbb{R}^n)} \le M_1 + M_1 r(\omega). \tag{6.7}$$

By invariance, $\mathcal{A}_\epsilon(\omega) = \Phi_\epsilon(t, \theta_{-t}\omega, \mathcal{A}_\epsilon(\theta_{-t}\omega))$ for all $t \ge 0$ and P-a.e. $\omega \in \Omega$. Therefore, by (6.7) we have that, for P-a.e. $\omega \in \Omega$,

$$\|u\|_{H^1(\mathbb{R}^n)} \le M_1 + M_1 r(\omega), \quad \forall\, u \in \bigcup_{0<\epsilon\le 1} \mathcal{A}_\epsilon(\omega). \tag{6.8}$$

We remark that (6.8) is important for proving the precompactness of the union $\bigcup\limits_{0<\epsilon\le 1}\mathcal{A}_\epsilon(\omega)$ in $L^2(\mathbb{R}^n)$.

Lemma 6.1. *Let $g\in L^2(\mathbb{R}^n)$ and* (4.8)-(4.11) *hold. Then the union $\bigcup\limits_{0<\epsilon\le 1}\mathcal{A}_\epsilon(\omega)$ is precompact in $L^2(\mathbb{R}^n)$.*

Proof. Given $\eta>0$, we want to show that the set $\bigcup\limits_{0<\epsilon\le 1}\mathcal{A}_\epsilon(\omega)$ has a finite covering of balls of radii less than η. Let R be a positive number and denote by

$$Q_R=\{x\in\mathbb{R}^n:\ |x|<R\}\quad\text{and}\quad Q_R^c=\mathbb{R}^n\setminus Q_R.$$

Let $\{K(\omega)\}_{\omega\in\Omega}$ be the random set given in (6.3). By Lemma 5.4, we find that, given $\eta>0$ and P-a.e. $\omega\in\Omega$, there exist $T=T(\omega,\eta)>0$ and $R=R(\omega,\eta)>0$ (independent of ϵ) such that for all $t\ge T$ and $u_0(\theta_{-t}\omega)\in K(\theta_{-t}\omega)$,

$$\int_{|x|\ge R}|u^\epsilon(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))(x)|^2dx\le\frac{\eta^2}{16}. \tag{6.9}$$

By (6.5), $u_0(\theta_{-t}\omega)\in\mathcal{A}_\epsilon(\theta_{-t}\omega)$ implies that $u_0(\theta_{-t}\omega)\in K(\theta_{-t}\omega)$. Therefore it follows from (6.9) that, for every $0<\epsilon\le 1$, P-a.e. $\omega\in\Omega$, $t\ge T$ and $u_0(\theta_{-t}\omega)\in\mathcal{A}_\epsilon(\theta_{-t}\omega)$,

$$\int_{|x|\ge R}|u^\epsilon(t,\theta_{-t}\omega,u_0(\theta_{-t}\omega))(x)|^2dx\le\frac{\eta^2}{16},$$

which along with the invariance of $\{\mathcal{A}_\epsilon(\omega)\}_{\omega\in\Omega}$ shows that, for P-a.e. $\omega\in\Omega$,

$$\int_{|x|\ge R}|u(x)|^2dx\le\frac{\eta^2}{16},\quad\forall\ u\in\bigcup_{0<\epsilon\le 1}\mathcal{A}_\epsilon(\omega),$$

that is for P-a.e. ω,

$$\|u\|_{L^2(Q_R^c)}\le\frac{\eta}{4},\quad\forall\ u\in\bigcup_{0<\epsilon\le 1}\mathcal{A}_\epsilon(\omega). \tag{6.10}$$

On the other hand, (6.8) implies that the set $\bigcup\limits_{0<\epsilon\le 1}\mathcal{A}_\epsilon(\omega)$ is bounded in $H^1(Q_R)$ for P-a.e. $\omega\in\Omega$. By the compactness of embedding $H^1(Q_R)\subseteq L^2(Q_R)$ we find that, for the given η, the set $\bigcup\limits_{0<\epsilon\le 1}\mathcal{A}_\epsilon(\omega)$ has a finite covering of balls of radii less than $\frac{\eta}{4}$ in $L^2(Q_R)$. This along with (6.10) shows that $\bigcup\limits_{0<\epsilon\le 1}\mathcal{A}_\epsilon(\omega)$ has a finite covering of balls of radii less than η in $L^2(\mathbb{R}^n)$. $\square$

Next, we investigate the limiting behavior of solutions of problem (4.1)-(4.2) when $\epsilon \to 0$. We further assume that the nonlinear function f satisfies, for all $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$,

$$|\frac{\partial f}{\partial s}(x,s)| \le \alpha_3 |s|^{p-2} + \psi_4(x), \tag{6.11}$$

where $\alpha_3 > 0$, $\psi_4 \in L^\infty(\mathbb{R}^n)$ if $p = 2$, and $\psi_4 \in L^{\frac{p}{p-2}}(\mathbb{R}^n)$ if $p > 2$.

Under condition (6.11), we will show that, as $\epsilon \to 0$, the solutions of the perturbed equation (4.1) converge to the limiting deterministic equation:

$$\frac{du}{dt} + \lambda u - \Delta u = f(x,u) + g(x), \quad x \in \mathbb{R}^n, \ t > 0. \tag{6.12}$$

Lemma 6.2. *Suppose $g \in L^2(\mathbb{R}^n)$, (4.8)-(4.11) and (6.11) hold. Given $0 < \epsilon \le 1$, let u^ϵ and u be the solutions of equation (4.1) and (6.12) with initial conditions u_0^ϵ and u_0, respectively. Then for P-a.e. $\omega \in \Omega$ and $t \ge 0$, we have*

$$\|u^\epsilon(t,\omega,u_0^\epsilon) - u(t,u_0)\|^2 \le ce^{ct}\|u_0^\epsilon - u_0\|^2 + \epsilon c e^{ct}\left(r(\omega) + \|u_0^\epsilon\|^2 + \|u_0\|^2\right),$$

where c is a positive deterministic constant independent of ϵ, and $r(\omega)$ is the tempered function in (4.4).

Proof. Let $v^\epsilon = u^\epsilon(t,\omega,u_0^\epsilon) - \epsilon z(\theta_t\omega)$ and $W = v^\epsilon - u$. Since v and u satisfy (4.7) and (6.12), respectively, we find that W is a solution of the equation:

$$\frac{\partial W}{\partial t} + \lambda W - \Delta W = f(x,u^\epsilon) - f(x,u) + \epsilon \Delta z(\theta_t\omega).$$

Taking the inner product of the above with W in $L^2(\mathbb{R}^n)$ we get

$$\frac{1}{2}\frac{d}{dt}\|W\|^2 + \lambda\|W\|^2 + \|\nabla W\|^2 = \int_{\mathbb{R}^n} (f(x,u^\epsilon) - f(x,u))W dx + \epsilon \int_{\mathbb{R}^n} \Delta z(\theta_t\omega) W dx. \tag{6.13}$$

For the first term on the right-hand side of (6.13), by (4.10) and (6.11) we have

$$\begin{aligned}\int_{\mathbb{R}^n} (f(x,u^\epsilon) - f(x,u))W dx &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial s}(x,s)(u^\epsilon - u) W dx \\ &= \int_{\mathbb{R}^n} \frac{\partial f}{\partial s}(x,s) W^2 dx + \epsilon \int_{\mathbb{R}^n} \frac{\partial f}{\partial s}(x,s) z(\theta_t\omega) W dx \\ &\le \beta\|W\|^2 + \epsilon\alpha_3 \int_{\mathbb{R}^n} (|u^\epsilon| + |u|)^{p-2} |z(\theta_t\omega)||W| dx + \epsilon \int_{\mathbb{R}^n} \psi_4 |z(\theta_t\omega)||W| dx\end{aligned}$$

$$\leq \beta\|W\|^2 + \epsilon c \left(\|u^\epsilon\|_p^p + \|u\|_p^p + \|z(\theta_t\omega)\|_p^p + \|W\|_p^p + \|\psi_4\|_{\frac{p}{p-2}}^{\frac{p}{p-2}} \right). \tag{6.14}$$

By the Young inequality, the last term on the right-hand side of (6.13) is bounded by

$$\epsilon \int_{\mathbb{R}^n} |\Delta z(\theta_t\omega) W| dx \leq \frac{1}{2}\epsilon\|\Delta z(\theta_t\omega)\|^2 + \frac{1}{2}\epsilon\|W\|^2 \leq \frac{1}{2}\epsilon\|\Delta z(\theta_t\omega)\|^2 + \frac{1}{2}\|W\|^2. \tag{6.15}$$

It follows from (6.13)-(6.15) that

$$\begin{aligned}\frac{d}{dt}\|W\|^2 &\leq c\|W\|^2 + \epsilon c + \epsilon c \left(\|u^\epsilon\|_p^p + \|u\|_p^p + \|z(\theta_t\omega)\|_p^p + \|\Delta z(\theta_t\omega)\|^2 + \|W\|_p^p \right) \\ &\leq c\|W\|^2 + \epsilon c + \epsilon c \left(\|u^\epsilon\|_p^p + \|u\|_p^p + \|z(\theta_t\omega)\|_p^p + \|\Delta z(\theta_t\omega)\|^2 \right) \\ &\leq c\|W\|^2 + \epsilon c + \epsilon c \left(\|u^\epsilon\|_p^p + \|u\|_p^p \right) + \epsilon c e^{\frac{1}{2}\lambda|t|} r(\omega),\end{aligned} \tag{6.16}$$

where we have used $W = u^\epsilon(t, \omega, u_0^\epsilon) - \epsilon z(\theta_t\omega) - u$, the fact $0 < \epsilon \leq 1$ and (4.6). Integrating (6.16) on (0,t) we obtain

$$\begin{aligned}\|W(t)\|^2 &\leq e^{ct}\|W(0)\|^2 + \epsilon c + \epsilon c r(\omega) e^{ct} \int_0^t e^{(\frac{1}{2}\lambda - c)s} ds \\ &\quad + \epsilon c \int_0^t e^{c(t-s)} \left(\|u^\epsilon(s, \omega, u_0^\epsilon)\|_p^p + \|u(s, u_0)\|_p^p \right) ds \\ &\leq e^{ct}\|W(0)\|^2 + \epsilon c_1 + \epsilon c_1 r(\omega) e^{c_2 t} + \epsilon c e^{ct} \int_0^t \left(\|u^\epsilon(s, \omega, u_0^\epsilon)\|_p^p + \|u(s, u_0)\|_p^p \right) ds.\end{aligned} \tag{6.17}$$

It follows from (5.8) that

$$\int_0^t e^{\lambda(s-t)} \|u^\epsilon(s, \omega, u_0^\epsilon)\|_p^p ds \leq e^{-\lambda t}\|v_0^\epsilon(\omega)\|^2 + \epsilon c \int_0^t e^{\lambda(s-t)} p_1(\theta_s\omega) ds + c,$$

which together with (5.6) implies that, for all $t \geq 0$,

$$\begin{aligned}\int_0^t e^{\lambda s} \|u^\epsilon(s, \omega, u_0^\epsilon)\|_p^p ds &\leq \|v_0^\epsilon(\omega)\|^2 + \epsilon c \int_0^t e^{\lambda s} p_1(\theta_s\omega) ds + c e^{\lambda t} \\ &\leq \|v_0^\epsilon(\omega)\|^2 + c r(\omega) \int_0^t e^{\frac{3}{2}\lambda s} ds + c e^{\lambda t} \leq \|u_0^\epsilon - \epsilon z(\omega)\|^2 + c_3 r(\omega) e^{c_4 t} + c e^{\lambda t}.\end{aligned} \tag{6.18}$$

Since $e^{\lambda s} \geq 1$ for all $s \in [0, t]$, we obtain from (6.18) that

$$\int_0^t \|u^\epsilon(s, \omega, u_0^\epsilon)\|_p^p ds \leq 2\|u_0^\epsilon\|^2 + 2\|z(\omega)\|^2 + c_3 r(\omega) e^{c_4 t} + c e^{\lambda t}. \tag{6.19}$$

Similarly, by (6.12) for $\epsilon = 0$, we can also get that

$$\int_0^t \|u(s, u_0)\|_p^p ds \leq c\|u_0\|^2 + c e^{\lambda t}. \tag{6.20}$$

By (4.4), (6.17) and (6.19)-(6.20) we find that, for all $t \geq 0$,

$$\|W(t)\|^2 \leq e^{ct}\|W(0)\|^2 + \epsilon c e^{c_5 t}\left(r(\omega) + \|u_0^\epsilon\|^2 + \|u_0\|^2\right). \tag{6.21}$$

Finally, by (4.6) and (6.21) we have, for all $t \geq 0$,

$$\|u^\epsilon(t,\omega,u_0^\epsilon) - u(t,u_0)\|^2 = \|W(t) + \epsilon z(\theta_t\omega)\|^2 \leq 2\|W(t)\|^2 + c_6\epsilon e^{c_7 t} r(\omega)$$

$$\leq 2e^{ct}\|W(0)\|^2 + \epsilon c e^{c_8 t}\left(r(\omega) + \|u_0^\epsilon\|^2 + \|u_0\|^2\right)$$

$$\leq 2e^{ct}\|u_0^\epsilon - u_0 - \epsilon z(\omega)\|^2 + \epsilon c e^{c_8 t}\left(r(\omega) + \|u_0^\epsilon\|^2 + \|u_0\|^2\right)$$

$$\leq 4e^{ct}\|u_0^\epsilon - u_0\|^2 + \epsilon c_9 e^{c_8 t}\left(r(\omega) + \|u_0^\epsilon\|^2 + \|u_0\|^2\right).$$

This completes the proof. $\square$

We are now in a position to establish the upper semicontinuity of the perturbed random attractors for problem (4.1)-(4.2).

Theorem 6.3. *Let $g \in L^2(\mathbb{R}^n)$, (4.8)-(4.11) and (6.11) hold. Then for P-a.e. $\omega \in \Omega$,*

$$\lim_{\epsilon \to 0} dist_{L^2(\mathbb{R}^n)}(\mathcal{A}_\epsilon(\omega), \mathcal{A}) = 0, \tag{6.22}$$

where

$$dist_{L^2(\mathbb{R}^n)}(\mathcal{A}_\epsilon(\omega), \mathcal{A}) = \sup_{a \in \mathcal{A}_\epsilon(\omega)} \inf_{b \in \mathcal{A}} \|a - b\|_{L^2(\mathbb{R}^n)}.$$

Proof. Note that $\{K_\epsilon(\omega)\}_{\omega\in\Omega}$ is a closed absorbing set for Φ_ϵ in $\mathcal{D}$, where $K_\epsilon(\omega)$ is given by (6.2). By (6.2) we find that

$$\limsup_{\epsilon \to 0} \|K_\epsilon(\epsilon)\| \leq M, \tag{6.23}$$

where M is the positive deterministic constant in (6.2). Let $\epsilon_n \to 0$ and $u_{0,n} \to u_0$ in $L^2(\mathbb{R}^n)$. Then by Lemma 6.2 we find that, for P-a.e. $\omega \in \Omega$ and $t \geq 0$,

$$\Phi_{\epsilon_n}(t,\omega,u_{0,n}) \to \Phi(t,u_0). \tag{6.24}$$

Notice that (6.23)-(6.24) and Lemma 6.1 indicate all conditions (3.1)-(3.3) are satisfied, and hence (6.22) follows from Theorem 3.1 immediately. $\square$

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